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COMMENT

Confinement properties for the Dirac equation with scalar-like and vector-like potentials

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Abstract. The (1+3)- and (1+1)-dimensional Dirac equation with both scalar-like and vector-like potentials is discussed. We prove that if the scalar-like potential is just equal to the vector-like potential, the confinement is impossible, i.e. there must be scattering states. Two exact solutions with linear potential and harmonic oscillator potential in this condition are given.

In a previous paper, the exact solutions of the Dirac equation with a linear scalar potential in a uniform electric field were given (Su and Zhang 1984). This example coincides with the general conclusion that if the scalar-like potential is stronger than the vector-like potential, the confinement is permanent and if, on the contrary, the vector-like potential is stronger, confinement is impossible due to the Klein paradox (Ni and Su 1980, Fishbane *et al* 1983, Long and Robson 1983). An interesting question is what would happen if the scalar-like potential were just equal to the vector-like one. In this comment, we would like to answer this question in general. We will prove that in the critical condition when the scalar-like potential in the Dirac equation is equal to the vector-like potential, the confinement is impossible, i.e. the scattering solution must exist.

The Dirac equation with both the scalar-like potential $\phi(r)$ and vector-like potential $V(r)$ is

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta(M+\phi)]\psi=(E-V)\psi. \qquad (1)$$

We can separate the angular part of equation (1) from the radial part by

$$\psi=\frac{1}{r}\begin{pmatrix} \mathrm{i}f(r)\varphi_{|\kappa|,m} \\ g(r)\varphi_{-|\kappa|,m}\end{pmatrix} \qquad (2)$$

where $\kappa=\mp(j+\frac{1}{2})$ and

$$\varphi_{|\kappa|,m}=\begin{pmatrix}-[(j-m+1)/(2j+2)]^{1/2}Y^{j+1/2}_{m-1/2} \\ [(j+m+1)/(2j+2)]^{1/2}Y^{j+1/2}_{m+1/2}\end{pmatrix} \qquad \varphi_{-|\kappa|,m}=\begin{pmatrix}[(j+m)/2j]^{1/2}Y^{j-1/2}_{m-1/2} \\ [(j-m)/2j]^{1/2}Y^{j-1/2}_{m+1/2}\end{pmatrix}. \qquad (3)$$

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The radial part of equation (1) is

$$\left[\mathrm{i}\sigma_2\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}\sigma_1+(M+\phi)\sigma_3\right]F=(E-V)F \tag{4}$$

where $F=\binom{f}{g}$, and

$$g'-\frac{\kappa}{r}g=(E-V-M-\phi)f \tag{5}$$

$$-f'-\frac{\kappa}{r}f=(E-V+M+\phi)g. \tag{6}$$

From equations (5) and (6), omitting the derivatives of ϕ and V, we obtain

$$f''+[(E^2-M^2)-2EV-2M\phi-(\phi^2-V^2)-\kappa(\kappa+1)/r^2]f=0. \tag{7}$$

Comparing it with the Schrödinger equation, we see that when the strengths of both potentials are large enough, the effective potential is (ϕ^2-V^2), i.e. the scalar-like potential $\phi(r)$ behaves like a barrier and the vector-like potential $V(r)$ like a well because they have different signs in the (ϕ^2-V^2) term. Obviously, if $\phi^2>V^2$ and ϕ goes to infinity at spatial infinity, there is an infinite barrier and confinement occurs; if $\phi^2<V^2$, there is an infinite well, and the Klein paradox occurs.

Now let us turn to the condition $V=\phi$. Equation (7) becomes

$$f''+[(E^2-M^2)-2(E+M)\phi-\kappa(\kappa+1)/r^2]f=0 \tag{8}$$

if there exists a scattering state solution for equation (8). We can find a quark at infinity in principle, which means there is no confinement for these potentials.

Suppose $\phi(r)=V(r)$ is a potential which tends to positive infinity at spatial infinity. Comparing equation (8) with the Schrödinger equation we see that there is an effective infinite barrier at spatial infinity for the positive-energy particles which will be responsible for the confinement solutions. However, there is an effective infinite well for the negative-energy particles which cannot prevent the negative-energy particles from going to infinity. In fact, if we choose the trial solution to be $f=c\,\mathrm{e}^{-\lambda r}$ in the region of r large enough, and substitute it into equation (8), we obtain the asymptotic equation

$$\lambda^2-2(E+M)\phi=0. \tag{9}$$

Therefore λ is real for the positive-energy particles, but imaginary for the negative energy particles. We obtain the bound state solutions for the positive-energy particles and the scattering state solutions for the negative-energy particles.

Similarly, when $\phi(r)=V(r)$ tends to negative infinity at spatial infinity, the positive-energy particles are not confined due to the effective potential well $2(E+M)\phi\to-\infty$.

In summary, we come to the conclusion that the confinement is impossible in the (1+3)-dimensional Dirac equation when the scalar-like potential is equal to the vector-like potential.

To clearly demonstrate this conclusion, let us discuss two soluble examples.

(*a*) $\phi=V=ar^2\ (a>0)$

This is the three-dimensional harmonic oscillator. Equation (8) becomes

$$f''+[k^2-\mu^2r^2-\kappa(\kappa+1)/r^2]f=0 \tag{10}$$

where

$$k^2 = E^2 - M^2 \tag{11}$$

$$\mu^2 = 2a(E+M). \tag{12}$$

(i) When $E+M>0$

The solutions of equation (10) are (Flügge 1974)

$$f = Cr^{\kappa+1}\,\mathrm{e}^{-\mu r^2/2}F(\alpha, \gamma, \mu r^2) \qquad \text{for } \kappa>0 \tag{13}$$

where

$$\alpha = \tfrac{1}{2}[\kappa + \tfrac{3}{2} - (k^2/2\mu)] \tag{14}$$

$$\gamma = \kappa + \tfrac{3}{2} \tag{15}$$

C is a normalised constant and $F(\alpha, \gamma, \mu r^2)$ is the Kummer function. For $\kappa>0$, solutions (13) are still correct except for exchanging κ for $(|\kappa|-1)$.

Another component of the spinor wavefunction can easily be found by equation (6):

$$g = -\frac{C}{E+M}r^{\kappa}\,\mathrm{e}^{-\mu r^2/2}\{[\kappa - \tfrac{1}{2} + (k^2/2\mu) - \mu r^2]F(\alpha, \gamma, \mu r^2)$$

$$+[\kappa + \tfrac{3}{2} - (k^2/2\mu)]F(\alpha+1, \gamma, \mu r^2)\}. \tag{16}$$

In order to avoid the exponentially divergent solutions at spatial infinity, we choose

$$\alpha = -n \qquad n = 0, 1, 2, \ldots$$

which will lead to energy quantisation. The asymptotic behaviours of f and g at spatial infinity are

$$\begin{aligned} f &\to r^{\kappa+1+2n}\,\mathrm{e}^{-\mu r^2/2} \\ g &\to r^{\kappa+2+2n}\,\mathrm{e}^{-\mu r^2/2}. \end{aligned} \tag{17}$$

This is a confinement solution.

(ii) When $E+M<0$

The solutions of equation (10) with $\mu = \mathrm{i}\lambda$ are

$$f = Dr^{\kappa+1}\,\mathrm{e}^{-\mathrm{i}\lambda r^2/2}F(\alpha', \gamma, \mathrm{i}\lambda r^2) \tag{18}$$

$$g = -\frac{D}{E+M}r^{\kappa}\,\mathrm{e}^{-\mathrm{i}\lambda r^2/2}\{[\kappa - \tfrac{1}{2} - (\mathrm{i}k^2/2\lambda) - \mathrm{i}\lambda r^2]F(\alpha', \gamma, \mathrm{i}\lambda r^2)$$

$$+[\kappa + \tfrac{3}{2} + (\mathrm{i}k^2/2\lambda)]F(\alpha'+1, \gamma, \mathrm{i}\lambda r^2)\} \tag{19}$$

where

$$\alpha' = \tfrac{1}{2}[\kappa + \tfrac{3}{2} + (\mathrm{i}k/2\lambda)]. \tag{20}$$

These solutions vanish at the origin but oscillate at spatial infinity:

$$f \to 2D\exp\left(-\frac{\pi k^2}{8\lambda} - \tfrac{1}{2}(\kappa + \tfrac{3}{2})\right)\left|\frac{\Gamma(\gamma)}{\Gamma(\alpha'^*)}\right|\frac{1}{\sqrt{r}}\cos\left(\frac{1}{2}\lambda r^2 + \frac{k^2}{2\lambda}\ln(\sqrt{\lambda}\,r) - \frac{\pi}{4}(\kappa + \tfrac{3}{2}) - \theta\right) \tag{21}$$

$$g \to \frac{2D}{E+M}\exp\left(-\frac{\pi k^2}{8\lambda} - \tfrac{1}{2}(\kappa - \tfrac{1}{2})\right)\left|\frac{\Gamma(\gamma)}{\Gamma(\alpha'^*)}\right|\sqrt{r}\sin\left(\frac{1}{2}\lambda r^2 + \frac{k^2}{2\lambda}\ln(\sqrt{\lambda}\,r) - \frac{\pi}{4}(\kappa + \tfrac{3}{2}) - \theta\right) \tag{22}$$

where

$$\theta = \arg[\Gamma(\gamma)/\Gamma(\alpha'^*)].$$

It means that the negative-energy particles cannot be confined.

(*b*) $\phi = V = br \ (b > 0)$

As the second example, let us discuss the linear potential. Equation (8) becomes

$$f'' + [(E^2 - M^2) - 2(E+M)br - \kappa(\kappa+1)/r^2]f = 0. \tag{23}$$

We will prove that for the negative-energy particles and $\kappa = -1$ there are scattering states, and thus the quark is not in confinement. In fact, when $\kappa = -1$ for a positive-energy particle $(E + M > 0)$ equation (23) becomes an Airy equation (Abramowitz and Stegun 1965) which has the form

$$(\mathrm{d}^2 f/\mathrm{d}y^2) - yf = 0 \tag{24}$$

where

$$y = [2b(E+M)]^{1/3} r - \frac{E^2 - M^2}{[2b(E+M)]^{2/3}}. \tag{25}$$

The solutions of equation (23) are the Airy functions

$$f = C\mathrm{Ai}(y) = C(y/3)^{1/2} K_{1/3}(\tfrac{2}{3}y^{3/2}) \qquad \text{for } y > 0 \tag{26}$$

$$f = C\mathrm{Ai}(-y) = \tfrac{1}{3}C(|y|)^{1/2}[J_{1/3}(\tfrac{2}{3}|y|^{3/2}) + J_{-1/3}(\tfrac{2}{3}|y|^{3/2})] \qquad \text{for } y < 0 \tag{27}$$

and the boundary condition at the origin

$$\mathrm{Ai}\left(\frac{E^2 - M^2}{[2b(E+M)]^{2/3}}\right) = 0 \tag{28}$$

gives us the energy eigenvalues. Equations (23) and (24) represent the bound state solutions. The function g can be obtained from equation (6) and we will not write it down explicitly. However, for the negative-energy particle, $E + M < 0$, in terms of the variable transformation

$$\tilde{y} = (2b|E+M|)^{1/3} r + \frac{E^2 - M^2}{(2b|E+M|)^{2/3}} \tag{29}$$

equation (23) becomes

$$(\mathrm{d}^2 f/\mathrm{d}\tilde{y}^2) + \tilde{y}f = 0 \tag{30}$$

and its solutions with the boundary condition that radial wavefunctions must vanish at the origin are

$$f = \sqrt{\tilde{y}}[C_1 J_{1/3}(\tfrac{2}{3}\tilde{y}^{3/2}) + C_2 J_{-1/3}(\tfrac{2}{3}\tilde{y}^{3/2})] \tag{31}$$

where

$$\frac{C_1}{C_2} = -\frac{J_{-1/3}(\tfrac{2}{3}\tilde{y}_0^{3/2})}{J_{1/3}(\tfrac{2}{3}\tilde{y}_0^{3/2})} \qquad \tilde{y}_0 = \frac{E^2 - M^2}{(2b|E+M|)^{2/3}}. \tag{32}$$

The asymptotic behaviours of f and g at spatial infinity are

$$f \to (3/\pi)^{1/2} \tilde{y}^{-1/4}[C_1 \cos(\tfrac{2}{3}\tilde{y}^{3/2} - \tfrac{5}{12}\pi) + C_2 \cos(\tfrac{2}{3}\tilde{y}^{3/2} - \tfrac{1}{12}\pi)] \tag{33}$$

$$g \to \frac{(2b|E+M|)^{1/3}}{E+M}\left(\frac{3}{\pi}\right)^{1/2} \tilde{y}^{1/4}[C_1 \sin(\tfrac{2}{3}\tilde{y}^{3/2} - \tfrac{5}{12}\pi) + C_2 \sin(\tfrac{2}{3}\tilde{y}^{3/2} - \tfrac{1}{12}\pi)]. \tag{34}$$

Obviously, this is an unconfinement solution so we come to the conclusion that the potential $\phi = V = br$ cannot confine quarks. Finally, we would like to point out that our conclusion can extend to $V = -\phi$. In fact, in this case equations (5) and (6) become

$$g'' + [(E - V)^2 - (M + \phi)^2 - \kappa(\kappa - 1)/r^2]g = 0. \tag{35}$$

We can use the same method for equation (8) to prove that the quark cannot be confined for $V = -\phi$.

In $(1+1)$ dimensions, the Dirac equation with the prescription $\alpha = \sigma_2$, $\beta = \sigma_1$ is

$$\left(-\mathrm{i}\sigma_2 \frac{\mathrm{d}}{\mathrm{d}x} + (M + \phi)\sigma_1\right)\begin{pmatrix} f \\ g \end{pmatrix} = (E - V)\begin{pmatrix} f \\ g \end{pmatrix} \tag{36}$$

$$-g' + (M + \phi)g = (E - V)f \tag{37}$$

$$f' + (M + \phi)f = (E - V)g. \tag{38}$$

Adding equation (37) to (38) and subtracting (37) from (38) we obtain

$$G' + (M + \phi - E - V)F = 0 \tag{39}$$

$$F' + (M + \phi + E - V)G = 0 \tag{40}$$

where $F = f + g$, $G = f - g$. Equations (39) and (40) have the same behaviours with equations (5) and (6) in the region for r large enough. The only difference is that the whole space of x replaces the half space of r. Then we can use the same method to reach the same conclusion that it cannot confine quarks for $\phi = \pm V$.

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